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The Optimization of a Functional within the Class of Solutions of a Partial Differential Equation

R. DELVER

Mathematical Centre, Amsterdam

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SUMMARY
This paper deals with a class of variational problems involving multiple integrals with one unknown function. Else than in the classical calculus of variations where the unknown function e.g. must be continuous and must take fixed values on the boundary, the unknown function must be a solution of a partial differential equation.

Physically one could imagine a process, described by a partial differential equation and controlled by the boundary conditions, while this process must be optimized in some sense by choosing the best boundary values.

1. Introduction

The calculus of variations deals in general with the optimization of a functional within some class of functions.

In the classical calculus of variations the class C_1 is often used [1].

In the optimal control theory the problem with the class of admissible functions consisting of the solutions of a set of ordinary differential equations, with given fixed boundary conditions, is taken up. In this case, the set of ordinary differential equations contains some control variables, that must be chosen in such a way that the functional reaches a relative extremum. A treatment of this problem is found in an article by R. Timman [3].

Extending this line we arrive at the subject of this paper. In sections 3 and 4, necessary conditions are derived, for a function $u(\xi^1, \ldots, \xi^n)$, which is a solution of a given linear elliptic partial differential equation with or without subsidiary conditions on the surface of the considered domain in R_n, to satisfy in order that this function attaches a relative extremum to the functional:

$$J\{u(\xi)\} = \int_G \cdots \int F(u, u_{\xi^1}, \ldots, u_{\xi^n}, \xi^1, \ldots, \xi^n) dG .$$

To optimize in this case means, to find boundary conditions so that the solution defined by this conditions and the differential equation, will attach a relative extremum to $J\{u(\xi)\}$. In section 5 the results of chapter 3 are given in the case of the Poisson equation.

This is done for some well-known coordinate systems.

Finally, some constructive applications will be found in sections 6 and 7.

2. Definitions

In this chapter some symbols are introduced, that will be used throughout this paper as defined here.

G, Γ	G is an open bounded domain in R_n with piecewise smooth surface Γ.
E_1	The set of all points of Γ where no normal vector exists. It is assumed that E_1 exists of a finite number of closed bounded domains in Γ with dimension less than $n-1$.
E_2	An arbitrary set of a finite number of closed, bounded domains in Γ with dimension less than $n-1$.

$\{x\}, \{\xi\}, T$ — $\{x\}$ Is a cartesian coordinate system in R_n.
$\{\xi\}$ Is a curvilinear coordinate system in R_n.
T is a reversible one-to-one transformation,
$T: x^i = x^i\{\xi^1, \ldots, \xi^n\} = x^i\{\xi\}$.

L, M — L is a second order linear partial differential operator in the cartesian coordinate system.

$$L = \left[a^{ik}(x) \frac{\partial^2}{\partial x^i \partial x^k} + b^i(x) \frac{\partial}{\partial x^i} + c(x) \right],$$

where: $a^{ik}(x) = a^{ki}(x)$; $a^{ik}(x) \in C_2\ [G \cup \Gamma]$,
$b^i(x) \in C_1\ [G \cup \Gamma]$,
$c\,(x) \in C\ [G \cup \Gamma]$.

It is assumed that the generalised Dirichlet problem for L and for its adjoint operator M has exactly one solution.

L', M' — The differential operators L and M, transformed for a coordinate system $\{\xi\}$.

C_1' — $u(\xi) \in C_1'$ means: $u(\xi) \in C_1\ [C\{E_2\} \cap \Gamma]$,
while $u(\xi)$, u_{ξ^i} are bounded for $\{\xi\} \in \Gamma$ $\quad (i = 1, 2, \ldots, n)$

U — $U: \{u(\xi): u(\xi) \in C_1' \cap L'\{u(\xi)\} = d(x(\xi)) \cap d(x) \in C[G \cup \Gamma]\}$.

$J\{u(\xi)\}$ — $J\{u(\xi)\} = \int_G \ldots \int F(u(\xi), u_{\xi^1}, \ldots, u_{\xi^n}, \xi^1, \ldots, \xi^n)\, dG$.

It is assumed that the integrand F has continuous first and second derivatives with respect to all its arguments.

$\hat{u}(\xi)$ — $\hat{u}(\xi)$ is a function $u(\xi) \in U$ which attaches a relative extremum to $J\{u(\xi)\}$.

3. General Theory

It is assumed that U contains a function $\hat{u}(\xi)$ which attaches a relative minimum to $J\{u(\xi)\}$. Necessary conditions, that must be satisfied by such a function will be derived.

For that purpose $\hat{u}(\xi)$ is varied within U with $\varepsilon\phi(\xi)$, ε is an arbitrary small real number and the varied function is called $\tilde{u}(\xi)$.

$$\tilde{u}(\xi) = \hat{u}(\xi) + \varepsilon\phi(\xi).$$

From $\tilde{u}(\xi) \in U$ follows:

$$L\{\phi(\xi)\} = 0, \quad \{\xi\} \in G, \tag{3.1}$$

and

$$\phi(\xi) \in C_1'. \tag{3.2}$$

From (3.1) and (3.2) follows that $\phi(\xi)$, $\{\xi\} \in G \cup \Gamma$, is bounded. Consequently $\varepsilon\phi(\xi)$ is an arbitrary small variation of $\hat{u}(\xi)$.

The increment of $J\{\hat{u}(\xi)\}$ is;

$$\begin{aligned} \Delta J\{\hat{u}(\xi); \varepsilon\phi(\xi)\} &= J\{\hat{u}(\xi) + \varepsilon\phi(\xi)\} - J\{\hat{u}(\xi)\}, \\ &= \int_G \ldots \int [F(\hat{u} + \varepsilon\phi, \hat{u}_\xi + \varepsilon\phi_\xi, \xi) - F(\hat{u}, \hat{u}_\xi, \xi)]\, dG, \\ &= \int_G \ldots \int [\varepsilon\{F_{\hat{u}}\phi + F_{\hat{u}_{\xi^j}}\phi_{\xi^j}\} + \varepsilon^2\{\ldots\} + \ldots]\, dG. \end{aligned}$$

The linear principle part or the variation of $J\{\hat{u}(\xi)\}$, $\delta J\{\hat{u}(\xi); \varepsilon\phi(\xi)\}$ is defined as:

$$\delta J\{\hat{u}(\xi); \varepsilon\phi(\xi)\} = \varepsilon \int_G \ldots \int \{F_{\hat{u}} \cdot \phi + F_{\hat{u}_{\xi^j}} \phi_{\xi^j}\}\, dG. \tag{3.3}$$

Hence:

$$\Delta J\{\hat{u}(\xi); \varepsilon\phi(\xi)\} = \delta J\{\hat{u}(\xi); \varepsilon\phi(\xi)\} + O(\varepsilon^2). \tag{3.4}$$

The condition that $\hat{u}(\xi)$ attaches a relative minimum to $J\{u(\xi)\}$ is equivalent to:

$$\Delta J\{\hat{u}(\xi)\,;\,\varepsilon\phi(\xi)\} \geqq 0\,, \tag{3.5}$$

for all functions $\phi(\xi)$ satisfying (3.1) and (3.2). This necessitates the following equation to hold good for all $\phi(\xi)$ according to (3.1) and (3.2):

$$\frac{1}{\varepsilon}\,\delta J\{\hat{u}(\xi)\,;\,\varepsilon\phi(\xi)\} = \int_G\!\!\ldots\!\int F_{\hat{u}}\cdot\phi + F_{\hat{u}_{\xi^j}}\cdot\phi_{\xi^j}\}\,dG = 0\,. \tag{3.6}$$

Via the divergence theorem of Gauss and the second theorem of Green, this integral will be transformed into an integral over Γ. Put:

$$\boldsymbol{F} = \begin{bmatrix} F^1 \\ \vdots \\ F^n \end{bmatrix},$$

$$F^i = \frac{\partial}{\partial u_{\xi^i}}\{F(u, u_{\xi^1}, \ldots, u_{\xi^n}, \xi^1, \ldots, \xi^n)\}\,.$$

$\boldsymbol{F}$ is a contravariant vector, so the divergence of $\boldsymbol{F}$ is:

$$\nabla\cdot\boldsymbol{F} = F^j{}_{,j} = \frac{\partial F^j}{\partial \xi^j} + \left\{ {j \atop kj} \right\} F^k\,,$$

hence:

$$\nabla\cdot\{\phi(\xi)\boldsymbol{F}\} - \phi(\xi)\{\nabla\cdot\boldsymbol{F}\} = \phi_{\xi^j}\cdot F^j\,.$$

Substitution into (3.6) yields:

$$\frac{1}{\varepsilon}\,\delta J\{\hat{u}(\xi)\,;\,\varepsilon\phi(\xi)\} = \int_G\!\!\ldots\!\int \{\phi(\xi)(F_u - \nabla\cdot\boldsymbol{F}) + \nabla\cdot(\phi(\xi)\boldsymbol{F})\}\,dG\,. \tag{3.7}$$

From the divergence theorem of Gauss follows:

$$\int_G\!\!\ldots\!\int \nabla\cdot\{\phi(\xi)\boldsymbol{F}\}\,dG = \oint\!\!\oint \phi(\xi)\{\boldsymbol{n}\cdot\boldsymbol{F}\}\,d\Gamma\,, \tag{3.8}$$

where $\boldsymbol{n}$ is the outward unit normal vector.

In cartesian coordinates the second theorem of Green exists. A variant of this theorem is:

$$\int_G\!\!\ldots\!\int \beta(x)\,M\{\alpha(x)\}\,dG = \oint\!\!\oint \beta(x)\,\frac{\partial\alpha(x)}{\partial\boldsymbol{v}}\,d\Gamma\,, \tag{3.9}$$

with the conditions:

$$\begin{array}{ll}
\alpha(x) \text{ and } \beta(x) \in C_2\,[G]\,, & \\
\dfrac{\partial\{\alpha(x)\}}{\partial\boldsymbol{v}} = \boldsymbol{v}\cdot\nabla\{\alpha(x)\}\,, & \{x\}\in\Gamma\,,\\
v^i(x) = a^{ik}(x)\,n_k(x)\,, & \{x\}\in\Gamma\,,\\
L\{\beta(x)\} = 0\,, & \{x\}\in G\,,\\
\alpha(x) = 0 \quad \text{for} & \{x\}\in\Gamma\,.
\end{array}$$

The transformation of (3.9) on the coordinate system $\{\xi\}$ results in:

$$\int_G\!\!\ldots\!\int \beta(x(\xi))\cdot M'\{\alpha(x(\xi)\}\,dG = \oint\!\!\oint \beta(x(\xi))\,\frac{\partial\{\alpha(x(\xi))\}}{\partial\boldsymbol{v}}\,d\Gamma\,, \tag{3.10}$$

where: $v^i = A^{\alpha\beta}(\xi)\,\dfrac{\partial x^i}{\partial\xi^\alpha}\,\dfrac{\partial x^k}{\partial\xi^\beta}\,n_k(x(\xi))\,,$

and: $A^{\alpha\beta}(\xi)$ are the coefficients of $\dfrac{\partial^2}{\partial\xi^\alpha \partial\xi^\beta}$ in L'.

The adjoint function $\psi(\xi)$ of a function $u(\xi)\in U$ is uniquely defined by:

$$\begin{cases} M'\{\psi(\xi)\} = F_u - \nabla\cdot \boldsymbol{F}, & \{\xi\}\in G, \\ \psi(\xi) = 0, & \{\xi\}\in \Gamma. \end{cases} \tag{3.11}$$

By virtue of (3.10), (3.1) and (3.11) it is found that:

$$\int_G \dots \int \phi(\xi)\{F_u - \nabla\cdot\boldsymbol{F}\}\,dG = \oint\!\!\oint \phi(\xi)\frac{\partial\{\psi(\xi)\}}{\partial \boldsymbol{v}}\,d\Gamma. \tag{3.12}$$

Substitution of (3.8) and (3.12) into (3.7) results in:

$$\frac{1}{\varepsilon}\,\delta J\{\hat{u}(\xi);\, \varepsilon\phi(\xi)\} = \oint\!\!\oint_\Gamma \phi(\xi)\left\{\boldsymbol{n}\cdot\boldsymbol{F} + \frac{\partial\{\psi(\xi)\}}{\partial\boldsymbol{v}}\right\}d\Gamma = 0. \tag{3.13}$$

This is the promised integral over Γ. The advantage of this expression is that $\phi(\xi)$ is an explicit factor in the integrand, with which the freedom of $\phi(\xi)$, for $\{\xi\}\in\Gamma$, can be utilized in a simple way. Put:

$$\tilde{\Gamma} = C\{E_1 \cup E_2\}\cap\Gamma.$$

The integrand of (3.13) does not exist on $E_1\cup E_2$. However, it can be easily concluded from the definitions that it is bounded on Γ and continuous on $\tilde{\Gamma}$.

Consequently, in (3.8), (3.12) and (3.13) Γ can be replaced by $\tilde{\Gamma}$.

The condition that (3.13) is satisfied for all functions $\phi(\xi)\in C_1'$ is equivalent to:

$$\boldsymbol{n}\cdot\boldsymbol{F} + \frac{\partial\{\psi(\xi)\}}{\partial\boldsymbol{v}} = 0, \quad \text{for} \quad \{\xi\}\in\tilde{\Gamma}.$$

Proof: $\tilde{\Gamma}$ is an open domain, consequently: at every point $\{\xi_0\}\in\tilde{\Gamma}$ exists a finite domain $A\{\xi_0\}$ with: $\{\xi_0\}\in A\{\xi_0\}\subset\tilde{\Gamma}$.

Suppose: $\boldsymbol{n}\cdot\boldsymbol{F} + \dfrac{\partial\{\psi(\xi)\}}{\partial\boldsymbol{v}} \neq 0$, say >0, for $\{\xi\}=\{\xi_1\}\in\tilde{\Gamma}$.

In that case an open domain $B\{\xi_1\}$ exists with: $\{\xi_1\}\in B\{\xi_1\}\subset A\{\xi_1\}$ while:

$$\boldsymbol{n}\cdot\boldsymbol{F} + \frac{\partial\{\psi(\xi)\}}{\partial\boldsymbol{v}} > 0, \quad \text{for} \quad \{\xi\}\in B\{\xi_1\}.$$

Let $\phi(\xi)\in C_1'$ satisfy:

$$\begin{cases} \phi(\xi)>0;\ \{\xi\}\in B\{\xi_1\}, \\ \phi(\xi)=0;\ \{\xi\}\in C\{B(\xi_1)\}\cap\Gamma. \end{cases}$$

This function however, attaches a positive value to (3.13), which is a contradiction. If

$\boldsymbol{n}\cdot\boldsymbol{F} + \dfrac{\partial\{\psi(\xi)\}}{\partial\boldsymbol{v}} = 0$, (3.13) is satisfied.

This completes the proof.

A function $u(\xi)\in U$ that satisfies (3.6) or (3.13) is called a stationary solution.

Now theorem I can be formulated:

Theorem I. *In order that $u(\xi)\in U$ is a stationary solution with regard to $J\{u(\xi)\}$, it is necessary and sufficient that $u(\xi)\in U$ has an adjoint function $\psi(\xi)$ which satisfies the overdetermined set of equations*:

$$\begin{cases} M'\{\psi(\xi)\} = F_u - \nabla\cdot\boldsymbol{F}\,, & \{\xi\}\in G\,, \\ \psi(\xi) = 0\,, & \{\xi\}\in \Gamma\,, \\ \boldsymbol{n}\cdot\boldsymbol{F} + \dfrac{\partial\{\psi(\xi)\}}{\partial\boldsymbol{v}} = 0\,, & \{\xi\}\in\tilde{\Gamma}\,. \end{cases} \tag{3.14}$$

There are still some remarks to be made:

(a) $\nabla\cdot\boldsymbol{F} = F^j_{,j} = \dfrac{1}{\sqrt{g}}\dfrac{\partial}{\partial\xi^j}\{\sqrt{g}\,F^j\}$

where: $g = \det(g^{ij})$
and: g^{ij} is the metric tensor.

(b) In order that $u(\xi)\in U$ has an adjoint function $\psi(\xi)$ it is necessary that:

$$\int_G\!\!\dots\!\int F_u\, dG = 0\,.$$

Proof:

$$\int_G\!\!\dots\!\int M'\{\psi(\xi)\}\,dG = \int_G\!\!\dots\!\int \{F_u - \nabla\cdot\boldsymbol{F}\}\,dG = \int_G\!\!\dots\!\int F_u\; dG - \oint_\Gamma \boldsymbol{n}\cdot\boldsymbol{F}\; d\Gamma\,.$$

On the other hand application of (3.10) with $\beta(\xi)\equiv 1$ leads to:

$$\int_G\!\!\dots\!\int M'\{\psi(\xi)\}\,dG = \oint_\Gamma \frac{\partial\{\psi(\xi)\}}{\partial\boldsymbol{v}}\,d\Gamma = -\oint_\Gamma \boldsymbol{n}\cdot\boldsymbol{F}\,d\Gamma\,.$$

This completes the proof.

(c) Though (3.14) has been derived with the use of the Dirichlet conditions, theorem I can also be used for other types of boundary value problems.

4. General Theory with Subsidiary Conditions on the Surface

It may happen that e.g. from technical considerations, the function $u(\xi)\in U$, $\{\xi\}\in\Gamma$, is not allowed to exceed some values or that this function has to take fixed values.

For this problem also, conditions can be derived which a function $\hat{u}(\xi)$ must satisfy.

In this case the class of admissible functions is:

$$W = \{u(\xi) : u(\xi)\in U \cap a(\xi) \leqq u(\xi) \leqq b(\xi),\ \{\xi\}\in\Gamma\}\,,$$

where $a(\xi)$ and $b(\xi)$ are two functions, defined on Γ, representing the upper and lower limitations of $u(\xi)$, $\{\xi\}\in\Gamma$, $a(\xi)\leqq b(\xi)$.

If a function $\hat{u}(\xi)$ is known, two domains Γ' and Γ'' can be distinguished on Γ:

Γ' : the set of all points $\{\xi\}\in\Gamma$, with $\hat{u}(\xi) = b(\xi)$,

Γ'': the set of all points $\{\xi\}\in\Gamma$, with $\hat{u}(\xi) = a(\xi)$.

With help of Γ' and Γ'' four domains α, β, γ and δ can be distinguished on Γ:

$$\begin{aligned} \alpha &= \{C(\Gamma'\cup\Gamma'')\}\cap\tilde{\Gamma}\,, \\ \beta &= \{C(\Gamma'')\}\cap\{\Gamma'\cap\tilde{\Gamma}\}\,, \\ \gamma &= \{C(\Gamma')\}\cap\{\Gamma''\cap\tilde{\Gamma}\}\,, \\ \delta &= \Gamma'\cap\Gamma''\,. \end{aligned} \tag{4.1}$$

As the varied function $\tilde{u}(\xi)$ must be an element of W, the variation $\varepsilon\phi(\xi)$ has to satisfy:

$$\begin{cases} L\{\phi(\xi)\} = 0\,, & \text{for} \quad \{\xi\}\in G\,, \\ \phi(\xi)\in C_1'\,, & \\ \varepsilon\phi(\xi) \begin{cases} \leqq 0\,, \\ \geqq 0\,, \\ =0\,, \end{cases} & \begin{matrix} \text{for} \quad \{\xi\}\in\beta\,, \\ \text{for} \quad \{\xi\}\in\gamma\,, \\ \text{for} \quad \{\xi\}\in\delta\,. \end{matrix} \end{cases} \tag{4.2}$$

The condition that $\hat{u}(\xi)$ attaches a relative minimum to $J\{u(\xi)\}$ is equivalent to:

$$\Delta J\{\hat{u}(\xi)\,;\,\varepsilon\phi(\xi)\} \geqq 0\,, \tag{4.3}$$

for all functions $\phi(\xi)$ satisfying (4.2). This necessitates the following equation to hold good for all $\varepsilon\phi(\xi)$, according to (4.2) with sufficiently small ε.

$$\delta J\{\hat{u}(\xi)\,;\,\varepsilon\phi(\xi)\} = \varepsilon \oint_\Gamma \phi(\xi)\left\{\boldsymbol{n}\cdot\boldsymbol{F} + \frac{\partial\{\psi(\xi)\}}{\partial\boldsymbol{v}}\right\} d\Gamma \geqq 0\,, \tag{4.4}$$

Analysis of (4.4), analogous to the analysis between (3.13) and theorem I, results in:

Theorem II. *In order that $u(\xi)\in W$ attaches a relative minimum to the functional $J\{u(\xi)\}$ it is necessary that its adjoint function satisfies the overdetermined set of equations*:

$$\begin{cases} M'\{\psi(\xi)\} = F_{\hat{u}} - \nabla\cdot\boldsymbol{F}\,, & \text{for} \quad \{\xi\}\in G\,, \\ \psi(\xi) = 0\,, & \text{for} \quad \{\xi\}\in\Gamma\,, \\ \boldsymbol{n}\cdot\boldsymbol{F} + \dfrac{\partial\{\psi(\xi)\}}{\partial\boldsymbol{v}} \begin{cases} =0\,, \\ \leqq 0\,, \\ \geqq 0\,, \end{cases} & \begin{matrix} \text{for} \quad \{\xi\}\in\bar{\alpha}\cap\tilde{\Gamma}\,, \\ \text{for} \quad \{\xi\}\in\beta\,, \\ \text{for} \quad \{\xi\}\in\gamma\,. \end{matrix} \end{cases} \tag{4.5}$$

In the case of a relative maximum the symbols α and β must be interchanged.

The domains α, β and γ will not be known until $\hat{u}(\xi)$ is known. This complicates a constructive application of theorem II.

5. The Equation of Poisson

In this chapter, the operator L is supposed to be the Laplace operator.

For this operator the notation Δ is used, both in the system $\{x\}$ and in the system $\{\xi\}$.

The Laplace operator is self-adjoint. Consequently:

$$\Delta = L = L' = M = M'\,.$$

In the case of the Laplace operator holds:

$$a^{ik}(x) = \delta^{ik}, \text{ consequently: } \boldsymbol{v} = \boldsymbol{n}\,.$$

The set of equations (3.14) takes the more simple form:

$$\begin{cases} \Delta\{\psi(\xi)\} = F_u - \nabla\cdot\boldsymbol{F}\,, & \{\xi\}\in G\,, \\ \psi(\xi) = 0\,, & \{\xi\}\in\Gamma\,, \\ \boldsymbol{n}\cdot\boldsymbol{F} + \dfrac{\partial\{\psi(\xi)\}}{\partial\boldsymbol{n}} = 0\,, & \{\xi\}\in\tilde{\Gamma}\,. \end{cases}$$

This set of equations will be specified for some well-known coordinate systems.

In the case of orthogonal coordinate systems can be applied:

$$\sqrt{g} = h_1\cdot h_2 \ldots h_n\,.$$

The scale factors can be deduced from the expression of the square infinitesimal arc length:

$$ds^2 = h_i^2 (d\xi^i)^2\,.$$

(a) *Cartesian coordinates in R_n*:
$\psi(x)$ must satisfy:

$$\begin{cases} \Delta\{\psi(x)\} = F_u - \sum_{j=1}^{n} \dfrac{\partial}{\partial x^j}\{F_{u_{x^j}}\}, & \{x\}\in G\,, \\ \psi(x) = 0\,, & \{x\}\in\Gamma\,, \\ \sum_{j=1}^{n} n_j\cdot\left\{F_{u_{x^j}} + \dfrac{\partial\{\psi(x)\}}{\partial x^j}\right\} = 0\,, & \{x\}\in\tilde{\Gamma}\,. \end{cases}$$

(b) *Cylindrical coordinates in R_3*:

$$\xi^1 = r \qquad ds^2 = dr^2 + r^2\,d\theta^2 + dz^2\,,$$
$$\xi^2 = \theta$$
$$\xi^3 = z \qquad \sqrt{g} = r\,,$$

$\psi(r,\theta,z)$ must satisfy:

$$\begin{cases} \Delta\{\psi(r,\theta,z)\} = F_u - \dfrac{\partial F_{u_r}}{\partial r} - \dfrac{\partial F_{u_\theta}}{\partial\theta} - \dfrac{\partial F_{u_z}}{\partial z} - \dfrac{1}{r}F_{u_r}\,, & \text{for } \{\xi\}\in G\,, \\ \psi(r,\theta,z) = 0\,, & \text{for } \{\xi\}\in\Gamma\,, \\ \boldsymbol{n}\cdot\begin{bmatrix} F_{u_r} \\ F_{u_\theta} \\ F_{u_z} \end{bmatrix} + \dfrac{\partial\psi(r,\theta,z)}{\partial\boldsymbol{n}} = 0\,, & \text{for } \{\xi\}\in\tilde{\Gamma}\,. \end{cases}$$

(c) *Spherical coordinates in R_3*:

$$\begin{cases} \xi^1 = r & ds^2 = dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\Phi \\ \xi^2 = \theta & \\ \xi^3 = \Phi & \sqrt{g} = r^2\sin\theta \end{cases}$$

ψ (r, θ, Φ) must satisfy:

$$\begin{cases} \Delta\{\psi(r,\theta,\phi) = F_u - \dfrac{\partial F_{u_r}}{\partial r} - \dfrac{\partial F_{u_\theta}}{\partial\theta} - \dfrac{\partial F_{u_\phi}}{\partial\phi} - \dfrac{1}{r}F_{u_r} - \operatorname{cotg}\theta\cdot F_{u_\theta}, & \{\xi\}\in G\,, \\ \psi(r,\theta,\phi) = 0\,, & \{\xi\}\in\Gamma\,, \\ \boldsymbol{n}\cdot\begin{bmatrix} F_{u_r} \\ F_{u_\theta} \\ F_{u_\phi} \end{bmatrix} + \dfrac{\partial\psi(r,\theta,\phi)}{\partial\boldsymbol{n}} = 0\,, & \{\xi\}\in\tilde{\Gamma}\,. \end{cases}$$

6. A Constructive Method

In this chapter a constructive method is developed for a collection of problems having the following restrictions:

- G is a circle domain with radius R.
- $L = \Delta = \left[\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right]$.
- $U = \{u(r,\theta): u(r,\theta)\in C_1' \cap \Delta\{u(r,\theta)\} = f(r,\theta) \cap f(r,\theta)\in C[G\cup\Gamma]\}$.
- $$\begin{aligned} F(u, u_r, u_\theta, r, \theta) = {} & a(r,\theta)u^2 + b(r,\theta)u_r^2 + c(r,\theta)u_\theta^2 + d(r,\theta)uu_r \\ & + e(r,\theta)uu_\theta + g(r,\theta)u_r u_\theta + h(r,\theta)u \\ & + j(r,\theta)u_r + k(r,\theta)u_\theta + l(r,\theta)\,, \end{aligned} \tag{6.1}$$

  with: $a(r,\theta), \ldots, l(r,\theta)\in C[G\cup\Gamma]$.

- A solution $u_0(r, \theta)$ of $\Delta\{u(r, \theta)\}$ can be found.
  This solution can be expanded in a twice term by term differentiable Fourier series:

$$u_0(r, \theta) = \frac{c_0(r)}{2} + \sum_{n=1}^{\infty} \{c_n(r)\cos(n\theta) + d_n(r)\sin(n\theta)\}\,. \tag{6.2}$$

- The adjoint function of any stationary solution can be expanded in a twice term by term differentiable Fourier series:

$$\psi(r, \theta) = x_0(r) + \sum_{n=1}^{\infty} \{x_n(r)\cos(n\theta) + y_n(r)\sin(n\theta)\}\,. \tag{6.3}$$

The Fourier expansion of an arbitrary harmonic function $w(r, \theta)$, defined on G is:

$$w(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n\cos(n\theta) + b_n\sin(n\theta)\}\left(\frac{r}{R}\right)^n$$

Consequently, the Fourier expansion of a function $u(r, \theta) \in U$ is:

$$u(r, \theta) = \frac{a_0 + c_0(r)}{2} + \sum_{n=1}^{\infty}\left\{a_n\left(\frac{r}{R}\right)^n + c_n(r)\right\}\cos(n\theta) + \sum_{n=1}^{\infty}\left\{b_n\left(\frac{r}{R}\right)^n + d_n(r)\right\}\sin(n\theta)\,. \tag{6.4}$$

In this expression only the constants $a_0, a_1, \ldots, b_1, b_2, \ldots$ are unknown.

At any stationary solution exactly one adjoint function $\psi(r, \theta)$ exists. This function satisfies:

$$\begin{cases} \Delta\{\psi(r, \theta)\} = F_u - \dfrac{\partial}{\partial r}\{F_{u_r}\} - \dfrac{\partial}{\partial \theta}\{F_{u_\theta}\} - \dfrac{1}{r}F_{u_r}\,, \\ \psi(R, \theta) = 0\,, \\ \dfrac{\partial\{\psi(R, \theta)\}}{\partial r} = -F_{u_r|R} \end{cases} \tag{6.5}$$

Substitution of (6.1) and (6.2) into (6.5) yields:

$$\begin{cases} \Delta\{\psi(r, \theta)\} = \alpha_0(r) + \beta_0(r)a_0 + \displaystyle\sum_{n=1}^{\infty} (\alpha_n(r) + \beta_n(r)a_n + \gamma_n(r)b_n)\cos(n\theta) \\ \qquad\qquad + \displaystyle\sum_{n=1}^{\infty} (\delta_n(r) + \varepsilon_n(r)a_n + \lambda_n(r)b_n)\sin(n\theta) \\ \psi(R, \theta) = 0\,, \\ \dfrac{\partial\psi(R, \theta)}{\partial r} = A_0 + B_0 a_0 + \displaystyle\sum_{n=1}^{\infty} (A_n + B_n a_n + C_n b_n)\cos(n\theta) \\ \qquad\qquad + \displaystyle\sum_{n=1}^{\infty} (D_n + E_n a_n + G_n b_n)\sin(n\theta)\,. \end{cases} \tag{6.6}$$

The functions $\alpha_0(r), \beta_0(r), \ldots, \lambda_n(r)$ and the constants $A_0, B_0, \ldots, G_n$ can be determined by performing the substitutions. In this chapter only the structure of (6.6) is of importance.

From the Fourier expansion (6.3) follows:

$$\begin{cases} \Delta\{\psi(r, \theta)\} = x_0''(r) + \dfrac{1}{r}x_0'(r) + \displaystyle\sum_{n=1}^{\infty} L_n\{x_n(r)\}\cos(n\theta) \\ \qquad\qquad + \displaystyle\sum_{n=1}^{\infty} L_n\{y_n(r)\}\sin(n\theta)\,, \\ \psi(R, \theta\} = x_0(R) + \displaystyle\sum_{n=1}^{\infty} \{x_n(R)\cos(n\theta) + y_n(R)\sin(n\theta)\}\,, \\ \dfrac{\partial\{\psi(R, \theta)\}}{\partial r} = x_0'(R) + \displaystyle\sum_{n=1}^{\infty} \{x_n'(R)\cos(n\theta) + y_n'(R)\sin(n\theta)\}\,, \end{cases} \tag{6.7}$$

where L_n is the differential operator:

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}\right].$$

The Fourier expansion is unique. From comparison of (6.6) with (6.7) follows (6.8), (6.9) and (6.10).

$$\begin{cases} x_0(R)=0\,, & \\ x_n(R)=0\,, & n=1,2,\dots\,, \\ y_n(R)=0\,, & n=1,2,\dots\,. \end{cases} \tag{6.8}$$

$$\begin{cases} x_0'(R)=A_0+B_0a_0\,, & \\ x_n'(R)=A_n+B_na_n+C_nb_n\,, & n=1,2,\dots\,, \\ y_n'(R)=D_n+E_na_n+G_nb_n\,, & n=1,2,\dots\,. \end{cases} \tag{6.9}$$

$$\begin{cases} x_0''(r)+x_0'(r)\cdot\frac{1}{r}=\alpha_0(r)+\beta_0(r)\cdot a_0\,, & \\ L_n\{x_n(r)\}=\alpha_n(r)+\beta_n(r)a_n+\gamma_n(r)b_n\,, & \text{for} \quad n=1,2,\dots\,, \\ L_n\{y_n(r)\}=\delta_n(r)+\varepsilon_n(r)a_n+\lambda_n(r)b_n\,, & \text{for} \quad n=1,2,\dots\,. \end{cases} \tag{6.10}$$

The solutions of (6.10) are:

$$\begin{cases} x_0(r)=K_0+N_0\ln(r)+P\{\alpha_0(r)\}+a_0\cdot P\{\beta_0(r)\}\,, \\ x_n(r)=K_nr^n+N_n\cdot r^{-n}+P\{\alpha_n(r)\}+a_n\cdot P\{\beta_n(r)\}+b_n\cdot P\{\gamma_n(r)\}\,, \\ y_n(r)=H_nr^n+M_nr^{-n}+P\{\delta_n(r)\}+a_n\cdot P\{\varepsilon_n(r)\}+b_n\cdot P\{\lambda_n(r)\}\,, \end{cases} \tag{6.11}$$

where $P_n\{\alpha_n(r)\}$ are particular solutions of $L_n\{x_n(r)\}=\alpha_n(r)$, etc. $\Delta\{\psi(r,\theta)\}$ has no singularities at the origin.

From this follows:

$$N_0=0\,,\quad N_n=0\,,\quad M_n=0\,.\qquad (n=1,2,\dots)$$

The unknown constants K_0, a_0, K_n, H_n, a_n and b_n ($n=1,2,\dots$), must be solved by imposing the boundary conditions (6.8) and (6.9) on (6.11). Substitution of (6.11) into (6.8) and (6.9) results for $n=0$, in one set of two linear equations in a_0 and K_0 and for $n=1$, $n=2$, etc. in one set of four linear equations in a_n, b_n, K_n and H_n.

With any stationary solution and its adjoint function a set of numbers a_0, a_n, K_0, K_n and H_n ($n=1,2,\dots$) exists, that has to satisfy the sets of linear equations.

On the other hand, any set of numbers a_0, a_n, b_n, K_0, K_n and H_n, that is a solution of the sets of linear equations leads by substitution of this values into (6.4) and into (6.11) and (6.3) respectively to a stationary solution and its adjoint function. Consequently, the method delivers exactly all stationary solutions.

Now three cases can be distinguished:

(1) One or more systems are contradictory.
Hence:
No stationary solution exists.

(2) One or more systems are dependent and no system is contradictory.
Hence:
There are an infinite number of stationary solutions.

(3) All systems deliver unique solutions for a_0, a_n, b_n, K_0, K_n and H_n ($n=1,2,\dots$).
Hence:
Exactly one stationary solution exists. This function is an extremum if an extremum is known to exist.

An application of this method is found in (7.2).

7. Applications

(1) Suppose:

– $J\{u(x)\} = \int_G \cdots \int \{au^2(x) + b(x)u(x) + c(x)\}\, dx^1 \ldots dx^n$,

where a is constant.

– $E_1 \cup E_2$ is empty.

– $L = \Delta = \left[\frac{\partial^2}{\partial x^{1^2}} + \ldots + \frac{\partial^2}{\partial x^{n^2}}\right]$.

An adjoint function must satisfy:

$$\begin{cases} \Delta\{\psi(x)\} = F - \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\{F_{u_{xj}}\} = 2au(x) + b(x), & \{x\} \in G, \\ \psi(x) = 0, \quad \{x\} \in \Gamma, & \\ \frac{\partial \psi(x)}{\partial \boldsymbol{n}} = -n_j \cdot F_{u_{xj}} = 0, \quad \{x\} \in \Gamma. & \end{cases} \tag{7.1}$$

All functions $u(x) \in U$ satisfy:

$$\Delta\{u(x)\} = d(x). \tag{7.2}$$

From (7.1) and (7.2) follows:

$$\begin{cases} \Delta\Delta\{\psi(x)\} = 2ad(x) + \Delta\{b(x)\}, & \{x\} \in G, \\ \psi(x) = 0, \ \{x\} \in T, & \\ \frac{\partial\{\psi(x)\}}{\partial \boldsymbol{n}} = 0, \ \{x\} \in T. & \end{cases} \tag{7.3}$$

This is a well-posed boundary value problem for the biharmonic equation. (7.3) has exactly one solution, which is the only possible adjoint function.

From $\Delta\{\psi(x)\} = 2au(x) + b(x)$ follows that exactly one stationary solution exists.

As the structure of $F(u(x), u_x, x)$ ensures the existence of a minimum, this minimum is:

$$\hat{u}(x) = \frac{\Delta\{\psi(x)\} - b(x)}{2a}.$$

This result can e.g. be used in the quadratic approximation within the class U of a function $v(x) \notin U$.

(2) This example is an application of the constructive method from chapter 6.
Suppose:

– $J\{u(r,\theta)\} = \int_0^R \int_0^{2\pi} \frac{1}{2}\{u^2(r,\theta) + u_r^2(r,\theta) + \frac{1}{r^2} u_\theta^2(r,\theta)\}\, r\, dr\, d\theta$.

– G is a circle domain with radius R.

– $E_1 \cup E_2$ is empty.

– $L = \Delta = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]$.

– $U = \{u(r,\theta);\ \Delta\{u(r,\theta)\} = 8r\sin\theta \cap u(r,\theta) \in C_1'\}$.

A solution of $\Delta\{u(r,\theta)\} = 8r\sin\theta$ is: $u_0(r,\theta) = r^3 \sin\theta$.

The Fourier expansion of a function $u(r,\theta) \in U$ is:

$$u(r, \theta) = \frac{a_0}{2} + r^3 \sin\theta + \sum_{n=1}^{\infty} \{a_n \cos(n\theta) + b_n \sin(n\theta)\} \left(\frac{r}{R}\right)^n . \tag{7.4}$$

With any stationary solution exists just one adjoint function $\psi(r, \theta)$ that satisfies:

$$\begin{cases} \Delta\{\psi(r, \theta) = F_u - \dfrac{\partial}{\partial r}\{F_{u_r}\} - \dfrac{\partial}{\partial\theta}\{F_{u_\theta}\} - \dfrac{1}{r}, \ F_{u_r} \\ \psi(R, \theta) = 0 , \\ \dfrac{\partial\{\psi(R, \theta)\}}{\partial r} = -F_{u_r|R} . \end{cases} \tag{7.5}$$

Or:

$$\begin{cases} \Delta\{\psi(r, \theta)\} = \dfrac{a_0}{2} + (r^3 - 8r)\sin\theta + \sum\limits_{n=1}^{\infty} \{a_n \cos(n\theta) + b_n \sin(n\theta)\} \left(\dfrac{r}{R}\right)^n , \\ \psi(R, \theta) = 0 , \\ \psi_r(R, \theta) = -3R^2 \sin\theta - \sum\limits_{n=1}^{\infty} \{a_n \cos(n\theta) + b_n \sin(n\theta)\} \dfrac{n}{R} . \end{cases} \tag{7.6}$$

From the direct Fourier expansion of $\psi(r, \theta)$ follows:

$$\Delta\{\psi(r, \theta)\} = x_0''(r) + \frac{1}{r} x'(r) + \sum_{n=1}^{\infty} L_n\{x_n(r)\} \cos(n\theta) + \sum_{n=1}^{\infty} L_n\{y_n(r)\} \sin(n\theta) ,$$

$$\psi(R, \theta) = x_0(R) + \sum_{n=1}^{\infty} \{x_n(R) \cos(n\theta) + y_n(R) \sin(n\theta)\} ,$$

$$\psi_r(R, \theta) = x_0'(R) + \sum_{n=1}^{\infty} \{x_n'(R) \cos(n\theta) + y_n'(R) \sin(n\theta)\} , \tag{7.7}$$

where $L_n = \left[\dfrac{d^2}{dr^2} + \dfrac{1}{r}\dfrac{d}{dr} - \dfrac{n^2}{r^2}\right]$.

From (7.6) and (7.7) follows:

$$\begin{cases} x_0''(r) + \dfrac{1}{r} x_0' = \dfrac{a_0}{2} , \\ L_n\{x_n(r)\} = a_n \left(\dfrac{r}{R}\right)^n , \qquad (n = 1, 2, 3, \ldots) , \\ L_1\{y_1(r)\} = r^3 - 8r + b_1 \dfrac{r}{R} , \\ L_n\{y_n(r)\} = b_n \left(\dfrac{r}{R}\right)^n , \qquad (n = 1, 2, 3, \ldots) , \end{cases} \tag{7.8}$$

with the solutions:

$$\begin{cases} x_0(r) = K_0 + N_0 \ln r + a_0 \left(\dfrac{r^2}{8}\right) , \\ x_n(r) = K_n r^n + N_n r^{-n} + a_n \left\{\dfrac{r^{n+2}}{4R^n(n+1)}\right\} , \qquad (n = 1, 2, \ldots) , \\ y_1(r) = H_1 r + M_1 r^{-1} + \frac{1}{24} r^5 - r^3 + b_1 \left(\dfrac{r^3}{8R}\right) , \\ y_n(r) = H_n r^n + M_n r^{-n} + b_n \left\{\dfrac{r^{n+2}}{4R^n(n+1)}\right\} , \qquad (n = 2, 3, \ldots) . \end{cases} \tag{7.9}$$

From $\Delta\{\psi(r,\theta)\}\in C[G]$ follows:

$$N_0=0,\quad N_n=0,\quad M_n=0,\qquad (n=1,2,\ldots).$$

Imposing the boundary conditions of (7.6) on (7.7) results in:

$$\begin{cases} x_0(R)=0\ ;\ x_0'(R)=0\,, \\ x_n(R)=0\ ;\ x_n'(R)=-a_n\cdot\dfrac{n}{R}\,, \qquad (n=1,2,\ldots)\,, \\ y_1(R)=0\ ;\ y_1'(R)=-b_n\cdot\dfrac{1}{R}-3R^2\,, \\ y_n(R)=0\ ;\ y_n'(R)=-b_n\cdot\dfrac{n}{R}\,, \qquad (n=2,3,\ldots)\,. \end{cases} \tag{7.10}$$

From (7.9) and (7.10) follows:

$$\begin{cases} a_n=0\,, \qquad (n=0,1,2,\ldots) \\ b_n=0\,, \qquad (n=1,2,3,\ldots) \\ b_1=-\dfrac{2R^5+12R^3}{3R^2+12}\,. \end{cases} \tag{7.11}$$

Substitution of these values in (7.4) yields:

$$\hat{u}(r,\theta)=\sin\theta\left\{r^3-rR^2\left[\frac{2R^2+12}{3R^2+12}\right]\right\}.$$

This function is generated by imposing the boundary function:

$$\hat{u}(R,\theta)=\sin\theta\left\{\frac{R^5}{3R^2+12}\right\}.$$

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